

# FORMAL ITERATED LOGARITHMS AND EXPONENTIALS AND THE STIRLING NUMBERS

THOMAS J. ROBINSON

**ABSTRACT.** We calculate the formal analytic expansions of certain formal translations in a space of formal iterated logarithmic and exponential variables. The results show how the algebraic structure naturally involves the Stirling numbers of the first and second kinds, and certain extensions of these, which appear as expansion coefficients.

## 1. INTRODUCTION

In [M] and [HLZ] logarithmic formal calculus was used to set up certain structures for the treatment of logarithmic intertwining operators and ultimately logarithmic tensor category theory for modules for a vertex operator algebra. A minor footnote in [HLZ] involved two expansions of certain formal series which yielded a classical combinatorial identity involving Stirling numbers of the first kind (see (3.17), [HLZ]), which those authors used to solve a problem posed in [Lu] (see Remark 3.8 in [HLZ]). These series expansions were worked out during the course of a proof of a logarithmic formal Taylor theorem (see Theorem 3.6, [HLZ]). A detailed treatment of an efficient algebraic method to obtain formal Taylor theorems in great generality was given in [R1]. This method was demonstrated on a space involving formal versions of iterated logarithmic and exponential variables, extending the setting used in [HLZ]. The method of proof bypasses any series expansions. The purpose of this paper is to calculate an arithmetically natural class of series expansions in this space of iterated logarithmic and exponential formal variables. We shall calculate two different expansions, each extending the methods used in the special case calculated in [HLZ]. Equating the coefficients of these expansions yields a class of combinatorial identities, including the special case found in [HLZ] that motivated this work. We shall also note the corresponding finite difference equations that certain of the expansion coefficients satisfy.

The combinatorial identity found in [HLZ] involves the Stirling numbers of the first kind. We shall find, from an algebraically symmetric setting, an analogous identity for the Stirling numbers of the second kind. We leave as an open question whether there is a simple or natural way to predict this fact from the algebra (beyond heuristics) or any salient features of this result having to do with relationships between the Stirling numbers of both types and also, conversely, whether or not the combinatorics might suggest any further natural generalization or extension of the algebraic setting.

This paper is almost entirely self-contained (in particular, no knowledge of vertex algebra theory, let alone the theory of logarithmic intertwining operators, is necessary to

read this paper). We need only one outside result, Theorem 6.1, which gives a recursion formula. We do develop from scratch the necessary definitions to fully understand the statement of this recursion, and refer the reader to [R2] for a full proof.

## 2. STIRLING NUMBERS AND OTHER SUNDRY ITEMS RECALLED

We recall in this section, and record the notation which we shall be using for, certain standard formal and combinatorial objects and facts which we use, or which appear during the course of this work.

For  $r \in \mathbb{C}$  and  $m \geq 0$  we extend, as usual, the binomial coefficient notation:

$$\binom{r}{m} = \frac{r(r-1) \cdots (r-m+1)}{m!}.$$

We make extensive use of the usual formal exponential and logarithmic series which we recall here.

$$(2.1) \quad e^{yX} = \sum_{n \geq 0} \frac{(yX)^n}{n!},$$

where  $y$  is a formal variable and where  $X$  is any formal object such that each coefficient may be finitely computed. Similarly

$$(2.2) \quad \log(1 + X) = \sum_{n \geq 0} \frac{(-1)^{n-1}}{n} X^n,$$

where, again,  $X$  is any formal object such that each coefficient may be finitely computed.

**Remark 2.1.** We shall also be using a second type of formal exponential and logarithm which we discuss in Section 4 (See also Remark 3.2 in [HLZ] where the presence of two different types of formal logarithm is discussed).

We shall be recovering certain classical facts about Stirling numbers. We review these well-known facts here (cf. [LW], [C]). Our results, among other things, show the equivalence of three different characterizations of the Stirling numbers of the first and second kinds. Each of these characterizations naturally arise during the course of the work in this paper. If one were working in ignorance of the classical results it would be natural to use different notation each time the Stirling numbers reappeared, and only at the end, when the equivalence is shown to unify the notation. However, since the results are classical, we shall anticipate these results and use standard notation throughout, unless otherwise indicated, for purposes of readability.

**Remark 2.2.** The Stirling numbers have combinatorial interpretations (cf. [LW], [C]) but we shall need only algebraic expressions for them here.

The (signless) Stirling numbers of the first kind, denoted by  $\begin{bmatrix} m \\ n \end{bmatrix}$  for  $0 \leq n \leq m$ , may be defined by either of the following two explicit expressions:

$$(2.3) \quad \begin{bmatrix} m \\ n \end{bmatrix} = \sum_{0 \leq t_1 < t_2 < \dots < t_{m-n} < m} t_1 \cdots t_{m-n}.$$

and

$$(2.4) \quad \begin{bmatrix} m \\ n \end{bmatrix} = \frac{m!}{n!} \sum_{\substack{i_1 + \dots + i_n = m \\ i_i \geq 1}} \frac{1}{i_1 \cdots i_n},$$

where in both (2.4) and (2.3),  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  is interpreted as 1 and  $\begin{bmatrix} m \\ 0 \end{bmatrix}$  is interpreted as 0 for  $m \geq 1$ .

We also recall, and it is routine to verify, using (2.4), that for  $n \geq 0$

$$(2.5) \quad (\log(1+x))^n = \sum_{m \geq n} (-1)^{m-n} \frac{n!}{m!} \begin{bmatrix} m \\ n \end{bmatrix} x^m.$$

The Stirling numbers of the first kind may also be defined as the unique solution to the following discrete boundary value problem.

$$(2.6) \quad \begin{bmatrix} m \\ n \end{bmatrix} = (m-1) \begin{bmatrix} m-1 \\ n \end{bmatrix} + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix},$$

for  $1 \leq n < m$  with

$$(2.7) \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 0 \quad \text{for } m > 0, \quad \text{and} \quad \begin{bmatrix} m \\ m \end{bmatrix} = 1 \quad \text{for } m \geq 0.$$

The Stirling numbers of the second kind, denoted by  $\left\{ \begin{smallmatrix} m \\ n \end{smallmatrix} \right\}$  for  $m, n \geq 0$ , may be defined by either of the following two explicit expressions:

$$(2.8) \quad \left\{ \begin{smallmatrix} m \\ n \end{smallmatrix} \right\} = \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{m-n} \leq n} i_1 i_2 \cdots i_{m-n}.$$

and

$$(2.9) \quad \left\{ \begin{smallmatrix} m \\ n \end{smallmatrix} \right\} = \frac{m!}{n!} \sum_{\substack{i_1 + i_2 + \dots + i_n = m \\ i_i \geq 1}} \frac{1}{i_1! i_2! \cdots i_n!},$$

where in both (2.9) and (2.8),  $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\}$  is interpreted as 1 and  $\left\{ \begin{smallmatrix} m \\ 0 \end{smallmatrix} \right\}$  is interpreted as 0 for  $m \geq 1$ .

We also recall, and it is routine to verify, using (2.9), that for  $n \geq 0$

$$(2.10) \quad (e^x - 1)^n = \sum_{m \geq n} \frac{n!}{m!} \left\{ \begin{smallmatrix} m \\ n \end{smallmatrix} \right\} x^m.$$

The Stirling numbers of the second kind may also be defined as the unique solution to the following discrete boundary value problem.

$$(2.11) \quad \left\{ \begin{matrix} m \\ n \end{matrix} \right\} = n \left\{ \begin{matrix} m-1 \\ n \end{matrix} \right\} + \left\{ \begin{matrix} m-1 \\ n-1 \end{matrix} \right\},$$

for  $1 \leq n < m$  with

$$(2.12) \quad \left\{ \begin{matrix} m \\ 0 \end{matrix} \right\} = 0 \quad \text{for } m > 0, \quad \text{and} \quad \left\{ \begin{matrix} m \\ m \end{matrix} \right\} = 1 \quad \text{for } m \geq 0.$$

### 3. THE AUTOMORPHISM PROPERTY

Our approach begins by considering a certain type of automorphism property. Let  $A$  be an algebra over  $\mathbb{C}$  and let  $D$  be a linear operator on  $A$ . We wish to consider iterating the action on  $D$  arbitrarily many times. Thus it is natural to consider a generating function. Let

$$f(x) = \sum_{n \geq 0} f_n x^n$$

where  $f_n \neq 0 \in \mathbb{C}$  and where  $x$  is a formal variable. We want the coefficients to be nonzero so that we account for every iteration of  $D$  (including the 0-th iteration) when considering the operator

$$f(xD) : A \rightarrow A[[x]],$$

where  $A[[x]]$  are the formal power series over  $A$ . Later we shall also use the notation  $A[x]$  to mean the formal polynomials over  $A$ . We take as a (logical) jumping off point the question of whether we can find  $f(x)$  to get

$$(3.1) \quad f(xD)(ab) = (f(xD)a)(f(xD)b), \quad (\text{the automorphism property})$$

for all  $a, b \in A$ .

**Remark 3.1.** This question is motivated by many well known such operators appearing in formal calculus, for instance, in Chapter 2 of [FLM]. Such an operator will turn out to be an analogue of an element of a one-parameter Lie group, but we shall not need any knowledge of such groups in this work.

The identity (3.1) consists of a sequence of identities, one for each power of  $x$ . The first two identities go as thus

$$\begin{aligned} f_0 ab &= f_0^2 ab \\ f_1 D(ab) &= f_1 f_0 (Da)b + f_0 f_1 a(Db). \end{aligned}$$

The first identity forces  $f_0 = 1$  since we insisted it be nonzero. Because  $f_1 \neq 0$  the second identity now gives us

$$D(ab) = (Da)b + a(Db),$$

or in other words, it shows that  $D$  must be a *derivation*. We may use this fact to expand  $D^m(ab)$  into terms of the form  $D^n a D^m b$  and because of this it is easily seen to be convenient (upon inspection after writing out the next few identities using the derivation expansion) to make the following assumption: given  $M, N \geq 0$  there exists  $a, b \in \mathbb{C}$  such that the elements  $D^n a D^m b$  for  $0 \leq n \leq N$  and  $0 \leq m \leq M$  are linearly independent. Then we are forced to equate coefficients in an easy fashion. In the  $n$ -th identity, we may, for instance, equate the coefficients of  $(D^{n-1}a)b$ . It is easy to see that this gives

$$n f_n = f_{n-1} f_1,$$

for  $n \geq 1$ . It is further easy to see that therefore

$$f_n = \frac{f_1^n}{n!},$$

for  $n \geq 1$ . Thus, in looking for an operator satisfying the automorphism property we are now reduced to the following possibility:

$$f(xD) = \sum_{n \geq 0} \frac{(f_1 x D)^n}{n!},$$

where  $D$  is a derivation. Recalling (2.1), we have

$$f(xD) = e^{f_1 x D}.$$

In fact, these operators do satisfy the automorphism property as we verify next.

**Proposition 3.1.** *(The “automorphism property”) Let  $A$  be an algebra over  $\mathbb{C}$  and let  $\alpha \in \mathbb{C}$ . Let  $D$  be a derivation on  $A$ . Then*

$$e^{\alpha y D}(ab) = (e^{\alpha y D}a) (e^{\alpha y D}b).$$

*Proof.* If  $\alpha = 0$  the statement is trivial. Expanding each side and equating coefficients leads to the identities

$$D^k(ab) = \sum_{\substack{n+m=k \\ n,m \geq 0}} \frac{k!}{n!m!} D^n a D^m b,$$

for  $k \geq 0$ . The  $k = 0$  identity is trivial and the  $k = 1$  identity is the derivation property of  $D$ . We proceed by induction assuming that we have verified the statement up to  $k$ .

Then

$$\begin{aligned}
D^{k+1}(ab) &= D(D^k(ab)) \\
&= D \sum_{\substack{n+m=k \\ n,m \geq 0}} \frac{k!}{n!m!} D^n a D^m b \\
&= \sum_{\substack{n+m=k \\ n,m \geq 0}} \frac{k!}{n!m!} D^{n+1} a D^m b + \sum_{\substack{n+m=k \\ n,m \geq 0}} \frac{k!}{n!m!} D^n a D^{m+1} b \\
&= \sum_{\substack{n+m=k+1 \\ n,m \geq 1}} \frac{k!}{(n-1)!m!} D^n a D^m b \\
&\quad + \sum_{\substack{n+m=k+1 \\ n,m \geq 1}} \frac{k!}{n!(m-1)!} D^n a D^m b + D^{k+1} a + D^{k+1} b \\
&= \sum_{\substack{n+m=k+1 \\ n,m \geq 1}} \left( \frac{k!}{(n-1)!m!} + \frac{k!}{n!(m-1)!} \right) D^n a D^m b + D^{k+1} a + D^{k+1} b \\
&= \sum_{\substack{n+m=k+1 \\ n,m \geq 0}} \frac{(k+1)!}{n!m!} D^n a D^m b.
\end{aligned}$$

□

We shall content ourselves with the case  $\alpha = 1$  for the remainder of the paper.

**Remark 3.2.** The preceding proposition is well known (cf. [LL]). It may be proved more easily if one assumes more information, for instance, about the binomial expansion (see e.g. the proof of Proposition 2.1 in [R1]). However, assuming this knowledge at this stage seems out of order philosophically as we shall essentially (re)-prove the binomial theorem as a special “base” case later on. Moreover, it is a philosophical point that it is “good” (vaguely “more intrinsic”) to take the automorphism property as primitive and later obtain combinatorial identities as “automatic” results, in this case arising from different calculations of certain formal expansion coefficients.

#### 4. FORMAL ITERATED LOGARITHMS AND EXPONENTIALS

Let  $\ell_n(x)$  be formal commuting variables for  $n \in \mathbb{Z}$ . We consider the algebra with an underlying vector space basis consisting of all elements of the form

$$\prod_{i \in \mathbb{Z}} \ell_i(x)^{r_i},$$

where  $r_i \in \mathbb{C}$  for all  $i \in \mathbb{Z}$ , and all but finitely many of the exponents  $r_i = 0$ . The multiplication is the obvious one (when multiplying two monomials simply add the corresponding exponents and linearly extend). We call this algebra

$$\mathbb{C}\{[\ell]\}.$$

We let  $\frac{d}{dx}$  be the unique derivation on  $\mathbb{C}\{[\ell]\}$  satisfying

$$\begin{aligned} \frac{d}{dx} \ell_{-n}(x)^r &= r \ell_{-n}(x)^{r-1} \prod_{i=-1}^{-n} \ell_i(x), \\ \frac{d}{dx} \ell_n(x)^r &= r \ell_n(x)^{r-1} \prod_{i=0}^{n-1} \ell_i(x)^{-1}, \\ \text{and} \quad \frac{d}{dx} \ell_0(x)^r &= r \ell_0(x)^{r-1}, \end{aligned}$$

for  $n > 0$  and  $r \in \mathbb{C}$ .

**Remark 4.1.** Secretly,  $\ell_n(x)$  is the  $(-n)$ -th iterated exponential for  $n < 0$  and the  $n$ -th iterated logarithm for  $n > 0$  and  $\ell_0(x)$  is  $x$  itself.

**Remark 4.2.** To see that this does indeed uniquely define a derivation, we note that  $\frac{d}{dx}$  must coincide with the unique linear map satisfying

$$\frac{d}{dx} \prod_{i \in \mathbb{Z}} \ell_i(x)^{r_i} = \sum_{j \in \mathbb{Z}} \frac{d}{dx} \ell_j(x)^{r_j} \prod_{i \neq j \in \mathbb{Z}} \ell_i(x)^{r_i},$$

on a basis of  $\mathbb{C}\{[\ell]\}$ . This establishes uniqueness. We need to check that this linear map is indeed a derivation. It is routine and we leave it to the reader to verify that it is enough to check that

$$\frac{d}{dx}(ab) = \left(\frac{d}{dx}a\right)b + a\left(\frac{d}{dx}b\right),$$

for basis elements  $a$  and  $b$ . Another routine calculation reduces the case to where  $a = \ell_i(x)^r$  and  $b = \ell_i(x)^s$  for  $r, s \in \mathbb{C}$ . Checking this case is trivial once one notes that

$$\frac{d}{dx} \ell_j(x)^r = r \ell_j(x)^{r-1} \frac{d}{dx} \ell_j(x).$$

If we let  $x$  and  $y$  be independent formal variables, then the formal exponentiated derivation  $e^{y \frac{d}{dx}}$ , defined by the expansion,  $\sum_{k \geq 0} y^k \left(\frac{d}{dx}\right)^k / k!$ , acts on a (complex) polynomial  $p(x)$  as a formal translation in  $y$ . That is, as the reader may easily verify, we have

$$(4.1) \quad e^{y \frac{d}{dx}} p(x) = p(x + y).$$

This motivates the following definition (as in [R1]).

**Definition 4.1.** *Let*

$$\ell_n(x+y) = e^{y\frac{d}{dx}}\ell_n(x) \quad \text{for } n \in \mathbb{Z}.$$

**Proposition 4.1.** *(The iterated exponential/logarithmic formal Taylor theorem)  
For  $p(x) \in \mathbb{C}\{[\ell]\}$  we have:*

$$e^{y\frac{d}{dx}}p(x) = p(x+y).$$

*Proof.* The result follows from the automorphism property. □

## 5. FORMAL ANALYTIC EXPANSIONS: METHOD 1

In this section we begin to calculate formal analytic expansions of  $\ell_N(x+y)^r$  for  $N \in \mathbb{Z}$ ,  $r \in \mathbb{C}$ . Recall Remark 4.1 for the “true meanings” of these objects. We shall begin with the cases when  $N \geq 0$ . We shall proceed step by step, first handling the cases  $N = 0$  and  $N = 1$  separately. The reader may skip ahead to Subsection 5.3 without any loss of generality.

**5.1. Case  $N = 0$ , the case “ $x$ ”.** It is easy to see how the  $m$ -th power of  $\frac{d}{dx}$  acts on  $\ell_0(x)^r$  because  $\frac{d}{dx}\ell_0(x)^r = r\ell_0(x)^{r-1}$ , which is a monomial. This is the essential observation that this method is based on, namely focusing on such “isolated” monomials. Later we shall have to expand  $\frac{d}{dx}$  as a sum of linear operators yielding such monomial results, but in this case it is already immediate that for  $r \in \mathbb{C}$  and  $m \geq 0$

$$\left(\frac{d}{dx}\right)^m \ell_0(x)^r = r(r-1)\cdots(r-m+1)\ell_0(x)^{r-m}.$$

**Proposition 5.1.** *For all  $r \in \mathbb{C}$*

$$e^{y\frac{d}{dx}}\ell_0(x)^r = \sum_{m \geq 0} \binom{r}{m} \ell_0(x)^{r-m} y^m.$$

□

**5.2. Case  $N = 1$ , the case “ $\log x$ ”.** We closely follow the argument leading to (3.15) in [HLZ]. We have for  $r, s \in \mathbb{C}$

$$\frac{d}{dx}\ell_0(x)^s \ell_1(x)^r = s\ell_0(x)^{s-1}\ell_1(x)^r + r\ell_0(x)^s \ell_1(x)^{r-1}.$$

Then define two linear operators  $T_0$  and  $T_1$  on  $\mathbb{C}[\ell_0(x), \ell_1(x)]$  by

$$\begin{aligned} T_0\ell_0(x)^s \ell_1(x)^r &= s\ell_0(x)^{s-1}\ell_1(x)^r \\ T_1\ell_0(x)^s \ell_1(x)^r &= r\ell_0(x)^s \ell_1(x)^{r-1}. \end{aligned}$$

Then

$$\left(\frac{d}{dx}\right)^m \ell_0(x)^s \ell_1(x)^r = (T_0 + T_1)^m \ell_0(x)^s \ell_1(x)^r.$$



It is not hard to see that

$$\left(\frac{d}{dx}\right)^m \ell_0(x)^s \ell_1(x)^r = \sum_{j=0}^m r(r-1)\cdots(r-j+1) \cdot \left( \sum_{0 \leq t_1 < t_2 < \cdots < t_{m-j} < m} (s-t_1)\cdots(s-t_{m-j}) \right) \ell_0(x)^{s-m} \ell_1(x)^{r-j},$$

where the reader should think of  $j$  as corresponding to the number of  $T_1$ 's in a summand of the expansion of  $(T_0 + T_1)^m$  and the  $t_i$ 's as corresponding to the positions of the  $T_0$ 's. Thus we have the following

**Proposition 5.2.** *For all  $r \in \mathbb{C}$*

$$(5.1) \quad e^{y \frac{d}{dx}} \ell_1(x)^r = \sum_{m \geq 0} \left( \frac{y}{\ell_0(x)} \right)^m \sum_{j=0}^m \binom{r}{j} \ell_1(x)^{r-j} \frac{j!}{m!} (-1)^{m-j} \cdot \left( \sum_{0 \leq t_1 < t_2 < \cdots < t_{m-j} < m} t_1 \cdots t_{m-j} \right).$$

□

**5.3. Case  $N \geq 0$ , “iterated logarithms”.** We begin by defining the following operators:

$$T_i = \prod_{j=0}^{i-1} \ell_j(x)^{-1} \frac{\partial}{\partial \ell_i(x)} \quad i \geq 0.$$

Then

$$\frac{d}{dx} = \sum_{i \geq 0} T_i.$$

Therefore,

$$(5.2) \quad \left(\frac{d}{dx}\right)^m = \sum_{i_1, i_2, \dots, i_m \geq 0} T_{i_1} T_{i_2} \cdots T_{i_m}.$$

If we consider all the monomials with a fixed number of occurrences of each  $T_i$ , and call this fixed number  $j_i$ , then we can partially calculate to get for  $c_l \in \mathbb{C}$

$$\left(\frac{d}{dx}\right)^m \prod_{l=0}^N (\ell_l(x))^{c_l} = \sum_{\substack{j_0 + j_1 + \cdots + j_N = m \\ 0 \leq j_0, j_1, \dots, j_N}} P(c) \prod_{i=0}^N \ell_i(x)^{c_i - \alpha_i},$$

where  $P(c)$  is a certain sum of polynomials in the  $c_i$  each of degree  $j_i$  in  $c_i$ , and where,  $\alpha_i$  is given by

$$(5.3) \quad \alpha_i = j_i + \cdots + j_N.$$

We shall describe  $P(c)$  by using a combinatorial construction, a type of tableau. A tableau will consist of a specified number of columns of blank entries each of a specified length. We shall construct a tableau on any such “grid” of blanks by filling in each blank with nonnegative numbers beginning at the top of each column and moving down. Each new entry can be any nonnegative number subject to two restrictions. First, the numbers must strictly ascend as one descends a column and second, the each entry must be less than or equal to the number of entries above and to the right (not necessarily above). So for example,

$$\begin{array}{ccc} & & 0 \\ & & 1 \\ 0 & & 2 \\ 2 & 5 & 3 \\ 8 & 6 & 4 \\ 11 & 7 & \end{array}$$

is a tableaux. But

$$\begin{array}{ccc} & & 0 \\ & & 1 \\ 0 & & 2 \\ 8 & 3 & 3 \\ 2 & 5 & 3 \\ 11 & 6 & 4 \end{array} \quad \text{and} \quad \begin{array}{ccc} & & 0 \\ & & 1 \\ & & 2 \\ 0 & 3 & 3 \\ 2 & 5 & 3 \\ 11 & 20 & 4 \end{array}$$

are not.

We shall consider all tableaux of a particular shape and assign to that shape a polynomial in as many variables,  $x_i$ , as there are columns. We shall denote this polynomial by

$$[m_1, m_2, \dots, m_n]_1(x_i),$$

where the shape is  $n$  columns of heights  $m_1$  on the left followed by  $m_2$  next to the right etc., where  $m_i \geq 0$ . The polynomial is found by summing over all the tableau of the given shape. Each summand is found by inserting “ $x_i$ —” in each entry of the  $i$ -th column (reading left to right) and multiplying all entries.

**Proposition 5.3.** *For all  $m \geq 0$ ,  $c_i \in \mathbb{C}$*

$$(5.4) \quad \left(\frac{d}{dx}\right)^m \prod_{l=0}^N (\ell_l(x))^{c_l} = \sum_{\substack{j_0+j_1+\dots+j_N=m \\ 0 \leq j_0, j_1, \dots, j_N}} [j_0, j_1, \dots, j_N]_1(c_i) \prod_{i=0}^N \ell_i(x)^{c_i - \alpha_i}.$$

*Proof.* It is easy to see that each summand of (5.2) when acting on  $\prod_{l=0}^N (\ell_l(x))^{c_l}$  will yield a coefficient corresponding to one of the tableaux. If one ‘constructs’ each term of (5.2) for a fixed  $j_0, \dots, j_N$  by first writing down all the  $T_N$ ’s and then inserting the  $T_{N-1}$ ’s from right to left etc. in all possible ways then it is easy to see the relevant one-to-one correspondence between the coefficients yielded by each term of (5.2) and the tableaux of a fixed shape given by the  $j_i$ ’s.  $\square$

Notice the way that writing down a term of (5.2) corresponded to filling in a tableau of a fixed shape. The labelling of each column depended only on the total number of entries there were in the columns to its right. That is, when constructing a tableau of fixed shape each column is labelled independently of the others. For instance, the rightmost column of a tableau is completely determined by its length alone. Therefore the piece of the ‘tableau polynomial’ due to the rightmost column may be factored out from all terms corresponding to a fixed tableau shape. If we look at the remaining factor of  $[m_1, m_2, \dots, m_n]_1(x_i)$  and set all the variables to zero, we get an integer which we shall call

$$(m_1, m_2, \dots, m_{n-1}; m_n)_1.$$

The independence of labeling gives immediately the following.

**Proposition 5.4.** *For  $m_1, \dots, m_n \geq 0$ ,*

$$(m_1, m_2, \dots, m_{n-1}; m_n)_1 = (m_1; m_2 + \dots + m_n)_1 \cdots (m_{n-2}; m_{n-1} + m_n)_1 (m_{n-1}; m_n)_1.$$

□

It is easy to calculate from the definition that

$$(m - j : j)_1 = (-1)^{m-j} \sum_{0 \leq t_1 < t_2 < \dots < t_{m-j} < m} t_1 \cdots t_{m-j}.$$

By (2.3) we have

$$(m - j : j)_1 = (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix}.$$

It is easy to see that we can specialize (5.4) to get

$$\begin{aligned} \left( \frac{d}{dx} \right)^m \ell_N(x)^r &= \sum_{\substack{j_0 + \dots + j_N = m \\ 0 \leq j_0, \dots, j_N}} j_N! \binom{r}{j_N} (j_0, \dots, j_{N-1}; j_N)_1 \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i} \\ &= \sum_{\substack{j_0 + \dots + j_N = m \\ 0 \leq j_0, \dots, j_N}} j_N! \binom{r}{j_N} \left( \prod_{i=0}^{N-1} (j_i; \alpha_{i+1})_1 \right) \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i} \\ &= \sum_{\substack{j_0 + \dots + j_N = m \\ 0 \leq j_0, \dots, j_N}} j_N! \binom{r}{j_N} (-1)^{\alpha_0 - \alpha_N} \left( \prod_{i=0}^{N-1} \begin{bmatrix} \alpha_i \\ \alpha_{i+1} \end{bmatrix} \right) \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i}. \end{aligned}$$

Thus we get

**Theorem 5.1.** *For  $r \in \mathbb{C}$ ,  $N \geq 0$ ,*

$$\ell_N(x + y)^r = \sum_{m \geq 0} \frac{y^m}{m!} \sum_{\substack{j_0 + j_1 + \dots + j_N = m \\ 0 \leq j_0, j_1, \dots, j_N}} j_N! \binom{r}{j_N} (-1)^{\alpha_0 - \alpha_N} \left( \prod_{i=0}^{N-1} \begin{bmatrix} \alpha_i \\ \alpha_{i+1} \end{bmatrix} \right) \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i}.$$

□

**5.4. Case  $N = -1$ , the case “exp  $x$ ”.** We shall next handle the cases  $N \leq 0$  step by step, first handling the cases  $N = -1$  and  $N = -2$  separately. The reader may skip ahead to Subsection 5.6 without any loss of generality.

For  $n, m \geq 0$  it is easy to see how the  $m$ -th power of  $\frac{d}{dx}$  acts on  $\ell_{-1}(x)^r$  because  $\frac{d}{dx}\ell_{-1}(x)^r = r\ell_{-1}(x)^{r-1}$  is a monomial. So we have  $\frac{d^m}{dx^m}\ell_{-1}(x)^r = r^m\ell_{-1}(x)^{r-m}$  which gives the following:

**Proposition 5.5.** *For all  $r \in \mathbb{C}$*

$$\begin{aligned} e^{y\frac{d}{dx}}\ell_{-1}(x)^r &= \sum_{m \geq 0} \frac{(ry)^m}{m!} \ell_{-1}(x)^{r-m} \\ &= \ell_{-1}(x)^r e^{ry}. \end{aligned}$$

□

**5.5. Case  $N = -2$ , the case “exp exp  $x$ ”.** We have for  $r, s \in \mathbb{C}$

$$\frac{d}{dx}\ell_{-1}(x)^s\ell_{-2}(x)^r = s\ell_{-1}(x)^{s-1}\ell_{-2}(x)^r + r\ell_{-1}(x)^s\ell_{-2}(x)^{r-1}.$$

Then define two linear operators  $S_0$  and  $S_1$  on  $\mathbb{C}[\ell_{-1}(x), \ell_{-2}(x)]$  by

$$\begin{aligned} S_0\ell_{-1}(x)^s\ell_{-2}(x)^r &= s\ell_{-1}(x)^{s-1}\ell_{-2}(x)^r \\ S_1\ell_{-1}(x)^s\ell_{-2}(x)^r &= r\ell_{-1}(x)^s\ell_{-2}(x)^{r-1}. \end{aligned}$$

Then

$$\left(\frac{d}{dx}\right)^m \ell_{-1}(x)^s\ell_{-2}(x)^r = (S_0 + S_1)^m \ell_{-1}(x)^s\ell_{-2}(x)^r.$$

It is not hard to see that

$$\begin{aligned} \left(\frac{d}{dx}\right)^m \ell_{-1}(x)^s\ell_{-2}(x)^r &= \sum_{j=0}^m r^j \left( \sum_{0 \leq t_1 \leq t_2 \leq \dots \leq t_{m-j} \leq j} (s+t_1) \cdots (s+t_{m-j}) \right) \\ &\quad \cdot \ell_{-1}(x)^{s+j}\ell_{-2}(x)^{r-j}, \end{aligned}$$

where the reader should think of  $j$  as corresponding to the number of  $S_1$ 's in a summand of the expansion of  $(S_0 + S_1)^m$  and the  $t_i$ 's as corresponding to the number of  $S_1$ 's to the right of the  $i$ -th  $S_0$ . It is now easy to get the following:

**Proposition 5.6.** *For all  $r \in \mathbb{C}$*

$$(5.5) \quad e^{y\frac{d}{dx}}\ell_{-2}(x)^r = \ell_{-2}(x)^r \sum_{m \geq 0} \frac{y^m}{m!} \sum_{j=0}^m (r\ell_{-1}(x))^j \sum_{0 \leq t_1 \leq t_2 \leq \dots \leq t_{m-j} \leq j} t_1 \cdots t_{m-j}.$$

□

**5.6. Formal analytic expansions: Cases  $N < 0$ , “iterated exponentials”.** We begin by defining the following operators:

$$T_i = \left( \prod_{j=1}^i \ell_{-j}(x) \right) \frac{\partial}{\partial \ell_{-i}(x)} \quad i > 0.$$

Then for our purposes

$$\frac{d}{dx} = \sum_{i>0} T_i.$$

Therefore,

$$\left( \frac{d}{dx} \right)^k = \sum_{i_1, i_2, \dots, i_k > 0} T_{i_1} T_{i_2} \cdots T_{i_k}.$$

If we consider all the monomials with a fixed number of occurrences of each  $T_i$ , and call this fixed number  $j_i$ , then we can partially calculate to get

$$\left( \frac{d}{dx} \right)^k \prod_{l=1}^N (\ell_{-l}(x))^{c_l} = \sum_{\substack{j_1 + \dots + j_N = k \\ 0 \leq j_1, \dots, j_N}} P(c) \ell_{-N}(x)^{c_N} \prod_{i=1}^{N-1} \ell_{-i}(x)^{c_i + \alpha_{i+1}},$$

where  $P(c)$  is a certain sum of polynomials in the  $c_i$  each of degree  $j_i$  in  $c_i$ , and where,  $\alpha_i$  is given by (5.3).

We shall describe  $P(c)$  by using a combinatorial construction, a type of tableau. A tableau will consist of a specified number of columns of blank entries each of a specified length. We shall construct a tableau on any such “grid” of blanks by filling in each blank with nonnegative numbers beginning at the top of each column and moving down. Each new entry can be any nonnegative number subject to two restrictions. First, the numbers must (non-strictly) ascend as one descends a column and second the entry in each column must be less than or equal to the number of entries to the right (not necessarily above). So for example,

$$\begin{array}{ccc} & & 0 \\ 0 & & 0 \\ 2 & 4 & 0 \\ 6 & 5 & 0 \\ 8 & 5 & 0 \end{array}$$

is a tableaux. But

$$\begin{array}{ccc} & 0 & \\ 0 & 0 & \\ 8 & 3 & 0 \\ 2 & 5 & 0 \\ 8 & 5 & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} & & 0 \\ & & 0 \\ 0 & 0 & \\ 2 & 3 & 0 \\ 8 & 5 & 0 \\ 8 & 6 & 0 \end{array}$$

are not.

We shall consider all tableaux of a particular shape and assign to that shape a polynomial in as many variables,  $x_i$ , as there are columns. We shall denote this polynomial by

$[m_1, m_2, \dots, m_n]_2(x_i)$  where the shape is  $n$  columns of heights  $m_1$  on the left followed by  $m_2$  next to the right etc. The polynomial is found by summing over all the tableau of the given shape. Each summand is found by inserting " $x_i$ " in each entry of the  $i$ -th column and multiplying all entries. Using similar reasoning to that in the proof of Proposition 5.3 we get

$$(5.6) \quad \left(\frac{d}{dx}\right)^k \prod_{l=1}^N (\ell_{-l}(x))^{c_l} = \sum_{\substack{j_1 + \dots + j_N = k \\ 0 \leq j_1, \dots, j_N}} [j_1, \dots, j_N]_2(c_i) \ell_{-N}(x)^{c_N} \prod_{i=1}^{N-1} \ell_{-i}(x)^{c_i + \alpha_{i+1}}.$$

We shall specialize this calculation, but shall first revisit the tableaux. Notice that the rightmost column of a tableau is completely determined by its length alone. Therefore the piece of the tableau polynomial due to the rightmost column may be factored out. If look at the remaining factor of  $[m_1, m_2, \dots, m_n]_2(x_i)$  and set all the variables to zero we get an integer which we shall call  $(m_1, m_2, \dots, m_{n-1}; m_n)_2$ . Notice that when constructing a tableau of fixed shape each column is labelled independently of the others. Thus we have

**Proposition 5.7.** *For  $m_1, \dots, m_n \geq 0$ ,*

$$(m_1, m_2, \dots, m_{n-1}; m_n)_2 = (m_1; m_2 + \dots + m_n)_2 \cdots (m_{n-2}; m_{n-1} + m_n)_2 (m_{n-1}; m_n)_2.$$

□

It is easy to see that

$$(m - n; n)_2 = \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_{m-n} \leq n} i_1 i_2 \cdots i_{m-n}.$$

By (2.8) we have,

$$(m - n; n)_2 = \left\{ \begin{matrix} m \\ n \end{matrix} \right\}.$$

Now we can specialize (5.6) to get for  $N \geq 0$

$$\begin{aligned} \left(\frac{d}{dx}\right)^k \ell_{-N}(x)^r &= \sum_{\substack{j_1 + j_2 + \dots + j_N = m \\ 0 \leq j_1, j_2, \dots, j_N}} r^{j_N} (j_1, \dots, j_{N-1}; j_N)_2 \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}} \\ &= \sum_{\substack{j_1 + \dots + j_N = m \\ 0 \leq j_1, j_2, \dots, j_N}} r^{j_N} \left( \prod_{i=1}^{N-1} (j_i; \alpha_{i+1})_2 \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}} \\ &= \sum_{\substack{j_1 + \dots + j_N = m \\ 0 \leq j_1, j_2, \dots, j_N}} r^{j_N} \left( \prod_{i=1}^{N-1} \left\{ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right\} \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}}. \end{aligned}$$

Thus we get

**Theorem 5.2.** For  $r \in \mathbb{C}$ ,  $N \geq 0$ ,

$$\ell_{-N}(x+y)^r = \sum_{m \geq 0} \frac{y^m}{m!} \sum_{\substack{j_1 + \dots + j_N = m \\ 0 \leq j_1, j_2, \dots, j_N}} r^{j_N} \left( \prod_{i=1}^{N-1} \left\{ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right\} \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}}.$$

□

## 6. FORMAL ANALYTIC EXPANSIONS: METHOD 2

In this section we begin to calculate formal analytic expansions of  $\ell_N(x+y)^r$  for  $N \in \mathbb{Z}$ ,  $r \in \mathbb{C}$  according to a second method. Recall Remark 4.1 for the “true meanings” of these objects. We shall begin with the “small  $N$ ” cases treating in order  $N = 0, 1, -1$  and  $-2$  separately. Then we shall deal with the general case. The reader may skip ahead to Section 7 without any loss of generality.

**6.1. Case  $N = 0$ , the case “ $x$ ”.** Here we first calculate  $e^{y \frac{d}{dx}} \ell_0(x)$  and then using the automorphism property, or alternatively the formal Taylor theorem, we may find  $e^{y \frac{d}{dx}} \ell_0(x)^n$  for  $n \in \mathbb{N}$  by using a (nonnegative integral) binomial expansion. We have

$$(6.1) \quad e^{y \frac{d}{dx}} \ell_0(x)^n = (\ell_0(x) + y)^n = \sum_{k \geq 0} \binom{n}{k} \ell_0(x)^{n-k} y^k.$$

We would like to extend this to the case where we have a general complex exponent. Of course, it is easy to do it directly, but we may also use the result for nonnegative integral exponents to help us, a method which is convenient in more difficult expansions.

**Proposition 6.1.** For all  $r \in \mathbb{C}$

$$e^{y \frac{d}{dx}} \ell_0(x)^r = (\ell_0(x) + y)^r = \sum_{m \geq 0} \binom{r}{m} \ell_0(x)^{r-m} y^m.$$

*Proof.* Notice that  $e^{y \frac{d}{dx}} \ell_0(x)^r \in \mathbb{C} \ell_0(x)^r [\ell_0(x)^{-1}][[y]]$  with coefficients being polynomials in  $r$ . When  $r$  is a nonnegative integer we have by (6.1) that the polynomial is

$$\binom{r}{m} = \frac{r(r-1) \cdots (r-m+1)}{m!},$$

and since polynomials are determined by a finite number of values we get the result. □

6.2. **Case  $N = 1$ , the case “log  $x$ ”.** We also have  $\left(\frac{d}{dx}\right)^l \ell_1(x) = (-1)^{l-1}(l-1)!\ell_0(x)^{-l}$  for  $l \geq 1$ . Recalling (2.2), we get that for  $n \in \mathbb{N}$ :

$$\begin{aligned}
 e^{y\frac{d}{dx}} \ell_1(x)^n &= \left(e^{y\frac{d}{dx}} \ell_1(x)\right)^n \\
 &= \left(\ell_1(x) + \log\left(1 + \frac{y}{\ell_0(x)}\right)\right)^n \\
 &= \sum_{j \geq 0} \binom{n}{j} \ell_1(x)^{n-j} \left(\sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \left(\frac{y}{\ell_0(x)}\right)^l\right)^j \\
 (6.2) \quad &= \sum_{j \geq 0} \binom{n}{j} \ell_1(x)^{n-j} \sum_{m \geq j} (-1)^{m-j} \sum_{\substack{m_1 + \dots + m_j = m \\ m_i \geq 1}} \frac{1}{m_1 \cdots m_j} \left(\frac{y}{\ell_0(x)}\right)^m.
 \end{aligned}$$

Then by arguing as in the proof of Proposition 6.1 we immediately get the following.

**Proposition 6.2.** *For all  $r \in \mathbb{C}$*

$$e^{y\frac{d}{dx}} \ell_1(x)^r = \sum_{j \geq 0} \binom{r}{j} \ell_1(x)^{r-j} \sum_{m \geq j} (-1)^{m-j} \sum_{\substack{m_1 + \dots + m_j = m \\ m_i \geq 1}} \frac{1}{m_1 \cdots m_j} \left(\frac{y}{\ell_0(x)}\right)^m.$$

□

6.3. **Case  $N = -1$ , the case “exp  $x$ ”.** We have for  $n \in \mathbb{N}$

$$\begin{aligned}
 e^{y\frac{d}{dx}} \ell_{-1}(x)^n &= \left(e^{y\frac{d}{dx}} \ell_{-1}(x)\right)^n \\
 &= (\ell_{-1}(x) e^y)^n \\
 &= \ell_{-1}(x)^n (1 + (e^y - 1))^n \\
 &= \ell_{-1}(x)^n \sum_{m \geq 0} \binom{n}{m} (e^y - 1)^m.
 \end{aligned}$$

Then by arguing as in the proof of Proposition 6.1 we immediately get the following.

**Proposition 6.3.** *For all  $r \in \mathbb{C}$*

$$e^{y\frac{d}{dx}} \ell_{-1}(x)^r = \ell_{-1}(x)^r \sum_{m \geq 0} \binom{r}{m} (e^y - 1)^m.$$

□



**6.4. Case  $N = -2$ , the case “exp exp  $x$ ”.** In order to proceed as in the previous three examples we would like to be able to easily calculate  $e^{y \frac{d}{dx}} \ell_{-2}(x)$  and then take the  $m$ -th power of the result. However, it is just as difficult to calculate  $e^{y \frac{d}{dx}} \ell_{-2}(x)$  as  $e^{y \frac{d}{dx}} \ell_{-2}(x)^m$ , since the answer already involves two variables non-trivially. We shall therefore use a different strategy which we could have used in place of method 4 in the case  $N = -1$  and also in a sense cases  $N = 0, 1$  although these cases are roughly like initial cases. We shall use a recursion formula which we state here.

**Theorem 6.1.** *For  $n \in \mathbb{Z}$  we have*

$$\ell_{n+1}(x+y) = \ell_{n+1}(x) + \log \left( 1 + \left( \frac{\ell_n(x+y) - \ell_n(x)}{\ell_n(x)} \right) \right)$$

and

$$\ell_n(x+y) = \ell_n(x) e^{(\ell_{n+1}(x+y) - \ell_{n+1}(x))}.$$

□

A proof of this is given in [R2].

## 7. FORMAL ANALYTIC EXPANSIONS: GENERAL CASE $N \in \mathbb{Z}$ “ITERATED LOGARITHMS AND EXPONENTIALS”

In this section we complete the calculation, using the second method, of the formal analytic expansion of  $\ell_N(x+y)^r$  for all  $N \in \mathbb{Z}$ .

**7.1. Iterated logarithms: second method.** Letting  $n \in \mathbb{N}$  and  $N \geq 0$ , we can use Theorem 6.1 to get:

$$\begin{aligned} \ell_N(x+y)^n &= \left( \ell_N(x) + \log \left( 1 + \left( \frac{\ell_{N-1}(x+y) - \ell_{N-1}(x)}{\ell_{N-1}(x)} \right) \right) \right)^n \\ &= \sum_{s \geq 0} \binom{n}{s} \ell_N(x)^{n-s} \cdot \left( \log \left( 1 + \ell_{N-1}(x)^{-1} \log \left( 1 + \left( \frac{\ell_{N-2}(x+y) - \ell_{N-2}(x)}{\ell_{N-2}(x)} \right) \right) \right) \right)^s. \end{aligned}$$

**Remark 7.1.** Compare with the calculation leading to (3.16) in [HLZ].

Iterating this and recalling (2.5), we get

$$\begin{aligned}
\ell_N(x+y)^n &= \sum_{0 \leq s_N} \binom{n}{s_N} \ell_N(x)^{n-s_N} \sum_{s_N \leq s_{N-1}} (-1)^{s_{N-1}-s_N} \frac{s_N!}{s_{N-1}!} \begin{bmatrix} s_{N-1} \\ s_N \end{bmatrix} \ell_{N-1}(x)^{-s_{N-1}} \\
&\quad \cdot \sum_{s_{N-1} \leq s_{N-2}} (-1)^{s_{N-2}-s_{N-1}} \frac{s_{N-1}!}{s_{N-2}!} \begin{bmatrix} s_{N-2} \\ s_{N-1} \end{bmatrix} \ell_{N-2}(x)^{-s_{N-2}} \dots \\
&\quad \dots \sum_{s_1 \leq s_0} (-1)^{s_0-s_1} \frac{s_1!}{s_0!} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \ell_0(x)^{-s_0} y^{s_0} \\
&= \ell_N(x)^n \sum_{0 \leq s_N \leq s_{N-1} \leq \dots \leq s_1 \leq s_0} \binom{n}{s_N} (-1)^{s_0-s_N} \frac{s_N!}{s_0!} \prod_{i=1}^N \begin{bmatrix} s_{i-1} \\ s_i \end{bmatrix} \prod_{i=0}^N \ell_i(x)^{-s_i} y_0^{s_0}.
\end{aligned}$$

Then by arguing as in the proof of Proposition 6.1 we immediately get the following.

**Theorem 7.1.** *For all  $r \in \mathbb{C}$ ,  $N \geq 0$*

$$\ell_N(x+y)^r = \ell_N(x)^r \sum_{0 \leq s_N < \dots < s_1 < s_0} \binom{r}{s_N} \frac{s_N!}{s_0!} \prod_{i=0}^{N-1} \begin{bmatrix} s_i \\ s_{i+1} \end{bmatrix} (-1)^{s_0-s_N} \prod_{i=0}^N \ell_i(x)^{-s_i} y_0^{s_0}.$$

□

**7.2. Iterated exponentials: second method.** We use (6.1) as in the last section, but this time the second form so that we may iterate in the other direction. We get for  $N \leq 0$  and  $n \in \mathbb{N}$

$$\begin{aligned}
\ell_N(x+y)^n &= \left( \ell_N(x) e^{(\ell_{N+1}(x+y) - \ell_{N+1}(x))} \right)^n \\
&= \ell_N(x)^n e^{n(\ell_{N+1}(x+y) - \ell_{N+1}(x))} \\
&= \ell_N(x)^n e^{-n\ell_{N+1}(x)} e^{n\ell_{N+1}(x+y)} \\
&= \ell_N(x)^n e^{-n\ell_{N+1}(x)} e^{n\ell_{N+1}(x) e^{(\ell_{N+2}(x+y) - \ell_{N+2}(x))}} \\
&= \ell_N(x)^n e^{n\ell_{N+1}(x) \left( e^{(\ell_{N+2}(x+y) - \ell_{N+2}(x))} - 1 \right)} \\
&= \ell_N(x)^n e^{n\ell_{N+1}(x) \left( e^{\ell_{N+2}(x) \left( e^{(\ell_{N+3}(x+y) - \ell_{N+3}(x))} - 1 \right)} - 1 \right)},
\end{aligned}$$

at which point the iteration is clear (the factor with  $n$  plays no role after the first iteration).

It is convenient to write  $N$  as a nonnegative number, so now, recalling (2.10), we let  $N \geq 0$  and  $n \in \mathbb{N}$  to get

$$\begin{aligned}
\ell_{-N}(x+y)^n &= \ell_{-N}(x)^n \sum_{0 \leq l_1} \frac{(n \ell_{-N+1}(x))^{l_1}}{l_1!} \\
&\quad \cdot \sum_{l_1 \leq l_2} \frac{\ell_{-N+2}(x)^{l_2} l_1!}{l_2!} \left\{ \begin{matrix} l_2 \\ l_1 \end{matrix} \right\} \sum_{l_2 \leq l_3} \frac{\ell_{-N+3}(x)^{l_3} l_2!}{l_3!} \left\{ \begin{matrix} l_3 \\ l_2 \end{matrix} \right\} \cdots \\
&\quad \sum_{l_{N-2} \leq l_{N-1}} \frac{\ell_{-1}(x)^{l_{N-1}} l_{N-2}!}{l_{N-1}!} \left\{ \begin{matrix} l_{N-1} \\ l_{N-2} \end{matrix} \right\} \sum_{l_{N-1} \leq l_N} \frac{y^{l_N} l_{N-1}!}{l_N!} \left\{ \begin{matrix} l_N \\ l_{N-1} \end{matrix} \right\} \\
&= \ell_{-N}(x)^n \sum_{0 \leq l_1 \leq l_2 \leq \cdots \leq l_N} n^{l_1} \left\{ \begin{matrix} l_2 \\ l_1 \end{matrix} \right\} \left\{ \begin{matrix} l_3 \\ l_2 \end{matrix} \right\} \cdots \left\{ \begin{matrix} l_{N-1} \\ l_{N-2} \end{matrix} \right\} \left\{ \begin{matrix} l_N \\ l_{N-1} \end{matrix} \right\} \\
&\quad \cdot \ell_{-N+1}(x)^{l_1} \ell_{-N+2}(x)^{l_2} \cdots \ell_{-1}(x)^{l_{N-1}} \frac{y^{l_N}}{l_N!}.
\end{aligned}$$

Then by arguing as in the proof of Proposition 6.1 we immediately get the following.

**Theorem 7.2.** *Let  $N \geq 0$  and  $r \in \mathbb{C}$  we get*

$$\ell_{-N}(x+y)^r = \ell_{-N}(x)^r \sum_{0 \leq l_1 \leq l_2 \leq \cdots \leq l_N} r^{l_1} \prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\} \prod_{i=1}^{N-1} \ell_{-i}(x)^{l_{N-i}} \frac{y^{l_N}}{l_N!}.$$

□

## 8. RECURRENCES

The coefficients of our various expansions satisfy linear recurrence relations. We shall indicate these recurrences for the low  $N$  cases giving the Stirling numbers of the first and second kinds and leave to the interested reader any routine generalization.

**8.1. Logarithmic case.** Let  $M(m, j)$  satisfy

$$\left( \frac{d}{dx} \right)^m \ell_0(x)^s \ell_1(x)^r = \sum_{j=0}^m r(r-1) \cdots (r-j+1) (-1)^{m-j} M(m, j) \ell_0(x)^{s-m} \ell_1(x)^{r-j}.$$

Two terms from  $\left( \frac{d}{dx} \right)^{m-1} \ell_0(x)^s \ell_1(x)^r$  contribute to each term of the next derivative. Namely

$$r(r-1) \cdots (r-j+1) (-1)^{m-1-j} M(m-1, j) \ell_0(x)^{s-m+1} \ell_1(x)^{r-j}$$

and

$$r(r-1) \cdots (r-j+2) (-1)^{m-j} M(m-1, j-1) \ell_0(x)^{s-m+1} \ell_1(x)^{r-j+1}$$

contribute to

$$r(r-1) \cdots (r-j+1) (-1)^{m-j} M(m, j) \ell_0(x)^{s-m} \ell_1(x)^{r-j},$$

which yields

$$\begin{aligned} r(r-1)\cdots(r-j+1)(-1)^{m-j}M(m,j) \\ = r(r-1)\cdots(r-j+1)(-1)^{m-1-j}(s-m+1)M(m-1,j) \\ + r(r-1)\cdots(r-j+2)(-1)^{m-j}(r-j+1)M(m-1,j-1), \end{aligned}$$

giving

$$M(m,j) = (m-1-s)M(m-1,j) + M(m-1,j-1),$$

for  $1 \leq j < m$  with boundary conditions easily seen to be given by

$$\begin{aligned} M(m,0) &= (-s)(1-s)\cdots(m-1-s) & m > 0, \\ \text{and } M(m,m) &= 1 & m \geq 0. \end{aligned}$$

This immediately gives the following.

**Proposition 8.1.** *For all  $r \in \mathbb{C}$*

$$e^{y \frac{d}{dx}} \ell_1(x)^r = \sum_{m \geq 0} \sum_{n=0}^m \binom{r}{n} \frac{n!}{m!} (-1)^{m-n} \begin{bmatrix} m \\ n \end{bmatrix} \left( \frac{y}{\ell_0(x)} \right)^m \ell_1(x)^{r-n},$$

where  $\begin{bmatrix} m \\ n \end{bmatrix}$  is given by (2.6) and (2.7).

**8.2. Exponential case.** Let  $N(m,j)$  satisfy

$$\left( \frac{d}{dx} \right)^m \ell_{-1}(x)^s \ell_{-2}(x)^r = \sum_{j \geq 0} r^j N(m,j) \ell_{-1}(x)^{s+j} \ell_{-2}(x)^r.$$

Two terms from  $\left( \frac{d}{dx} \right)^{m-1} \ell_{-1}(x)^s \ell_{-2}(x)^r$  contribute to each term of the next derivative. Namely

$$r^j N(m-1,j) \ell_{-1}(x)^{s+j} \ell_{-2}(x)^r$$

and

$$r^{j-1} N(m-1,j-1) \ell_{-1}(x)^{s+j-1} \ell_{-2}(x)^r$$

each contribute to

$$r^j N(m,j) \ell_{-1}(x)^{s+j} \ell_{-2}(x)^r,$$

which yields

$$r^j N(m,j) = r^j (s+j) N(m-1,j) + r^{j-1} r N(m-1,j-1),$$

giving

$$N(m,j) = (s+j) N(m-1,j) + N(m-1,j-1),$$

for  $1 \leq j < m$  with boundary conditions obviously given by

$$\begin{aligned} N(m, m) &= 1 & m \geq 0 \\ \text{and } N(m, 0) &= s^m & m > 0. \end{aligned}$$

This immediately gives the following.

**Proposition 8.2.** *For all  $r \in \mathbb{C}$*

$$e^{y \frac{d}{dx}} \ell_{-2}(x)^r = \sum_{m \geq 0} \sum_{n=0}^m \frac{r^n}{m!} \left\{ \begin{matrix} m \\ n \end{matrix} \right\} \ell_{-1}(x)^n \ell_{-2}(x)^r y^m,$$

where  $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}$  is given by (2.11) and (2.12).

## 9. IDENTITIES

Because we used two different methods to calculate the formal analytic expansions we may equate the results to get combinatorial identities. We have been anticipating some of these results already and so have already used the same notation for two different expressions for the Stirling numbers. Therefore, temporarily, in this section, Stirling numbers of the first kind will be denoted by  $\left[ \begin{matrix} m \\ n \end{matrix} \right]_1$  when given by (2.3) and by  $\left[ \begin{matrix} m \\ n \end{matrix} \right]_2$  when given by (2.4). Similarly, Stirling numbers of the second kind will be denoted by  $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}_1$  when given by (2.8) and by  $\left\{ \begin{matrix} m \\ n \end{matrix} \right\}_2$  when given by (2.9).

**9.1. Logarithmic case.** We wish to equate the expansions of Theorems 5.1 and 7.1. We have for  $r \in \mathbb{C}$  and  $N \geq 0$ ,

$$\begin{aligned} \ell_N(x+y)^r &= \sum_{m \geq 0} \frac{y^m}{m!} \sum_{\substack{j_0+j_1+\dots+j_N=m \\ 0 \leq j_0, j_1, \dots, j_N}} j_N! \binom{r}{j_N} (-1)^{\alpha_0 - \alpha_N} \left( \prod_{i=0}^{N-1} \left[ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right]_1 \right) \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i} \\ &= \ell_N(x)^r \sum_{0 \leq s_N < \dots < s_1 < s_0} \binom{r}{s_N} \frac{s_N!}{s_0!} \prod_{i=0}^{N-1} \left[ \begin{matrix} s_i \\ s_{i+1} \end{matrix} \right]_2 (-1)^{s_0 + s_N} \prod_{i=0}^N \ell_i(x)^{-s_i} y^{s_0}. \end{aligned}$$

Recalling that

$$\alpha_i = j_i + \dots + j_N,$$

we get

$$j_i = \alpha_i - \alpha_{i+1},$$

for  $0 \leq i \leq N-1$  and  $\alpha_N = j_N$ . Then we may rewrite the first expression for  $\ell_N(x+y)^r$  as

$$\ell_N(x+y)^r = \sum_{0 \leq \alpha_N \leq \dots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0} \alpha_N! \binom{r}{\alpha_N} (-1)^{\alpha_0 - \alpha_N} \left( \prod_{i=0}^{N-1} \left[ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right]_1 \right) \ell_N(x)^r \prod_{i=0}^N \ell_i(x)^{-\alpha_i} \frac{y^{\alpha_0}}{\alpha_0!},$$

and now equating coefficients, we get

$$\prod_{i=0}^{N-1} \begin{bmatrix} s_i \\ s_{i+1} \end{bmatrix}_1 = \prod_{i=0}^{N-1} \begin{bmatrix} s_i \\ s_{i+1} \end{bmatrix}_2,$$

which of course, is only interesting in the  $N = 1$  case, which gives the classical identity:

$$\frac{m!}{n!} \sum_{\substack{i_1 + \dots + i_n = m \\ i_l \geq 1}} \frac{1}{i_1 \dots i_n} = \sum_{0 \leq t_1 < t_2 < \dots < t_{m-n} < m} t_1 \dots t_{m-n},$$

for  $1 \leq n \leq m$ . Well actually, it gives more than this, because it gives an automatic proof of Proposition 5.4. That is, equating coefficients also gives

$$(\alpha_0 - \alpha_1, \dots, \alpha_{N-1} - \alpha_N; \alpha_N)_1 = \prod_{i=0}^{N-1} \begin{bmatrix} \alpha_i \\ \alpha_{i+1} \end{bmatrix}_2.$$

Then the  $N = 1$  case gives

$$(\alpha_0 - \alpha_1; \alpha_1)_1 = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}_2,$$

which yields

$$(\alpha_0 - \alpha_1, \dots, \alpha_{N-1} - \alpha_N; \alpha_N)_1 = \prod_{i=0}^{N-1} (\alpha_i - \alpha_{i+1}; \alpha_{i+1})_1,$$

or

$$(j_0, \dots, j_{N-1}; j_N)_1 = \prod_{i=0}^{N-1} (j_i; j_{i+1} + \dots + j_N)_1,$$

which is Proposition 5.4.

**9.2. Exponential case.** We wish to equate the expansions of Theorems 5.2 and 7.2. We have for  $r \in \mathbb{C}$  and  $N \geq 0$ ,

$$\begin{aligned} \ell_{-N}(x+y)^r &= \sum_{m \geq 0} \frac{y^m}{m!} \sum_{\substack{j_1 + \dots + j_N = m \\ 0 \leq j_1, j_2, \dots, j_N}} r^{j_N} \left( \prod_{i=1}^{N-1} \begin{bmatrix} \alpha_i \\ \alpha_{i+1} \end{bmatrix}_1 \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}} \\ &= \ell_{-N}(x)^r \sum_{0 \leq l_1 \leq l_2 \leq \dots \leq l_N} r^{l_1} \prod_{i=1}^{N-1} \begin{bmatrix} l_{i+1} \\ l_i \end{bmatrix}_2 \prod_{i=1}^{N-1} \ell_{-i}(x)^{l_{N-i}} \frac{y^{l_N}}{l_N!}. \end{aligned}$$

As in the last section, we may substitute  $\alpha_i$ 's for the  $j_i$ 's to get

$$\begin{aligned} & \sum_{0 \leq \alpha_N \leq \dots \leq \alpha_2 \leq \alpha_1} \frac{y^{\alpha_1}}{\alpha_1!} r^{\alpha_N} \left( \prod_{i=1}^{N-1} \left\{ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right\}_1 \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{\alpha_{i+1}} \\ &= \ell_{-N}(x)^r \sum_{0 \leq l_1 \leq l_2 \leq \dots \leq l_N} r^{l_1} \prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\}_2 \prod_{i=1}^{N-1} \ell_{-i}(x)^{l_{N-i}} \frac{y^{l_N}}{l_N!}, \end{aligned}$$

which gives

$$\begin{aligned} & \sum_{0 \leq l_1 \leq \dots \leq l_{N-1} \leq l_N} \frac{y^{l_N}}{l_N!} r^{l_1} \left( \prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\}_1 \right) \ell_{-N}(x)^r \prod_{i=1}^{N-1} \ell_{-i}(x)^{l_{N-i}} \\ &= \ell_{-N}(x)^r \sum_{0 \leq l_1 \leq l_2 \leq \dots \leq l_N} r^{l_1} \prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\}_2 \prod_{i=1}^{N-1} \ell_{-i}(x)^{l_{N-i}} \frac{y^{l_N}}{l_N!}, \end{aligned}$$

yielding

$$\prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\}_1 = \prod_{i=1}^{N-1} \left\{ \begin{matrix} l_{i+1} \\ l_i \end{matrix} \right\}_2,$$

which, of course, is only interesting in the  $N = 2$  case, which gives the classical identity:

$$\frac{m!}{n!} \sum_{i_1+i_2+\dots+i_n=m} \frac{1}{i_1!i_2!\dots i_n!} = \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n} i_1 i_2 \dots i_m,$$

for  $1 \leq n \leq m$ . Well, actually, as in the logarithm case, it gives more, because it gives an automatic proof of Proposition 5.7. That is, equating coefficients also gives

$$(\alpha_1 - \alpha_2, \dots, \alpha_{N-1} - \alpha_N; \alpha_N)_2 = \prod_{i=1}^{N-1} \left\{ \begin{matrix} \alpha_i \\ \alpha_{i+1} \end{matrix} \right\}_2.$$

Then the  $N = 2$  case gives

$$(\alpha_1 - \alpha_2; \alpha_2)_2 = \left\{ \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} \right\}_2,$$

which yields

$$(\alpha_1 - \alpha_2, \dots, \alpha_{N-1} - \alpha_N; \alpha_N)_2 = \prod_{i=1}^{N-1} (\alpha_i - \alpha_{i+1}; \alpha_{i+1})_2,$$

or

$$(j_1, \dots, j_{N-1}; j_N)_2 = \prod_{i=1}^{N-1} (j_i; j_{i+1} + \dots + j_N)_2,$$

which is Proposition 5.7.

## REFERENCES

- [C] D. Cohen, *Basic Techniques of Combinatorial Theory*, John Wiley & Sons, New York, 1978.
- [FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, New York, 1988.
- [HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, arXiv:0710.2687v3 [math.QA]
- [LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.
- [LW] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1992.
- [Lu] D. Lubell, Problem 10992, Problems and Solutions, *American Mathematical Monthly* **110**, 2003, 155.
- [M] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in: *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201–225.
- [R1] T. Robinson, New perspectives on exponentiated derivations, the formal Taylor theorem, and Faà di Bruno's formula, in: *Proceedings of the Conference on Vertex Operator Algebras, Illinois State University*, (2008) ed. by: Maarten Bergvelt, Gaywalee Yamskulna, and Wenhua Zhao, Contemporary Math. **497**, Amer. Math. Soc., Providence (2009), 185–198; arXiv:0903.3391.
- [R2] T. Robinson, A recursion identity for formal iterated logarithms and iterated exponentials, arXiv:1012.0822.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854

*E-mail address*: thomasro@math.rutgers.edu